# Higher-Order Radiative Corrections to Bhabha Scattering at Low Angles: The YFS Monte Carlo Approach*

S. Jadach[1,2], M. Melles[3], W. Płaczek[4], E. Richter-Wąs[2,4], M. Skrzypek[3], B.F.L. Ward[3], Z. Wąs[2,1] and S. Yost[3]

[1] Institute of Nuclear Physics, Kraków, Poland

[2] CERN, Theory Division, Geneva, Switzerland

[3] Department of Physics and Astronomy, University of Tennessee, Knoxville, Tennessee, USA

[4] Institute of Computer Science, Jagellonian University, Kraków, Poland

**Abstract**

In this contribution we present new numerical results for the QED second-order radiative corrections to the low-angle Bhabha cross-section from the new Monte Carlo event generator, BHLUMI4.0. Our main concern is the precision of these calculations. We discuss quantitatively both technical precision — numerical problems — and physical precision — higher orders. The results presented here will be essential in the future reduction of the overall theoretical uncertainty in the measurement of the luminosity at LEP below the present 0.25% level.

emails:
jadach@cernvm.cern.ch, mike@hephp01.phys.utk.edu, placzek@hephp01.phys.utk.edu, erichter@cernvm.cern.ch, skrzypek@hpjmiady.ifj.edu.pl, bflw@slacvm.slac.stanford.edu, wasm@cernvm.cern.ch, syost@hephp01.phys.utk.edu

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# 1 Basics and notation

Luminosity at LEP/SLC is measured with the help of the low-angle Bhabha (LABH) process, $e^+e^- \to e^+e^-$, in the angular range below 100 mrad. The luminosity determined in this way provides absolute normalization of the cross-section of all other processes in the $e^+e^-$ scattering. The LABH cross-section is therefore not of physical interest in itself. On the contrary, it is regarded as completely known from theory, from Quantum Electrodynamics (QED). However, although in principle the LABH cross-section is calculable in perturbative QED with arbitrary precision (except for a small hadronic correction), it is subject to theoretical uncertainties due to a truncation of the perturbative expansion and to a limitation of the calculational tools (computer programs). All LEP/SLC experiments use theoretical calculations for LABH, based on works published by some of the present authors three years ago [1]. This calculation has an overall theoretical/technical precision of 0.25% and is embodied in the form of the Monte Carlo event generator [2] BHLUMI version 2.0. This error was acceptable in 1991 but now, with an improvement in the experimental precision of a factor of two or more, it dominates the present overall luminosity error. It is therefore quite urgent to reduce the theoretical error of the QED calculation to a precision level of at least 0.1%.

The backbone of the 0.25% theoretical precision estimate [1] is due to missing second-order $\mathcal{O}(\alpha^2L^2)$ — 0.15% — and $\mathcal{O}(\alpha^2L)$ — 0.09% — contributions in the matrix element encoded in the Monte Carlo calculations. Here $L = \ln(|t|/m_e^2)$ is the so-called big log in the leading-logarithmic (LL) approximation where $t$ is the $t$-channel transfer — of order 1 GeV; see Fig. 1 for a pictorial definition of the LL approximation.

The first of the above contributions (0.15%) also includes the technical precision of the Monte Carlo programs due to programming bugs, rounding errors, quality of random numbers, etc. It is illustrated in Fig. 2, taken from Ref. [1] as a difference of three Monte Carlo subgenerators of BHLUMI 2.0. These are: (i) multiphoton $\mathcal{O}(\alpha)_{\rm exp}$ BHLUMI, (ii) $\mathcal{O}(\alpha)$ OLDBIS (without exponentiation) and, (iii) $\mathcal{O}(\alpha^3)_{\rm LL}$, the leading-logarithmic (collinear photon emission) event subgenerator LUMLOG. The differences in the three Monte Carlo subgenerators provide a solid estimate of the technical precision. In addition, subgenerators (ii) and (iii) have separate estimates of their technical precision at the level below 0.05%, coming in the case of (ii) from an independent comparison [3] with a semianalytical calculation, and in the case of (iii) with another Monte Carlo [4]. The comparison in Fig. 2 therefore provides an estimate of the technical precision mainly for the multiphoton BHLUMI subgenerator, which did not have any other independent analytical or Monte Carlo cross-check[1].

As for the physical precision, which is mainly due to a truncation of the perturbative calculation [1], the dominant (beyond first-order) correction of $\mathcal{O}(\alpha^2L^2)$ was under good control because it was calculated using the $\mathcal{O}(\alpha^3)_{\rm LL}$ subgenerator LUMLOG [4]. The hybrid Monte Carlo calculation OLDBIS plus LUMLOG includes the entire $\mathcal{O}(\alpha^2L^2)$ correction for the integrated cross-section, but due to the zero-angle collinear emission of photons, LUMLOG is not very suitable for experimental analysis where various fine-grain inclusive/multiphoton distributions are checked in the process of reducing the systematic

[1] At the time it seemed unthinkable to analytically integrate the total cross-section of the $\mathcal{O}(\alpha)_{\rm exp}$ BHLUMI.

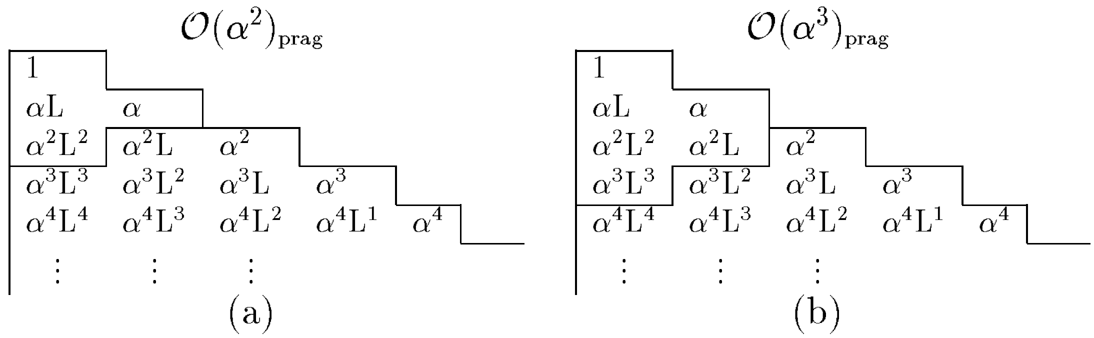


Figure 1: QED perturbative leading and subleading corrections. Rows represent corrections in consecutive perturbative orders — the first row is the Born contribution. The first column represents the leading logarithmic (LL) approximation and the second the next-to-leading (NLL) approximation. Terms selected for (a) second- and (b) third-order pragmatic expansion are limited by an additional line.

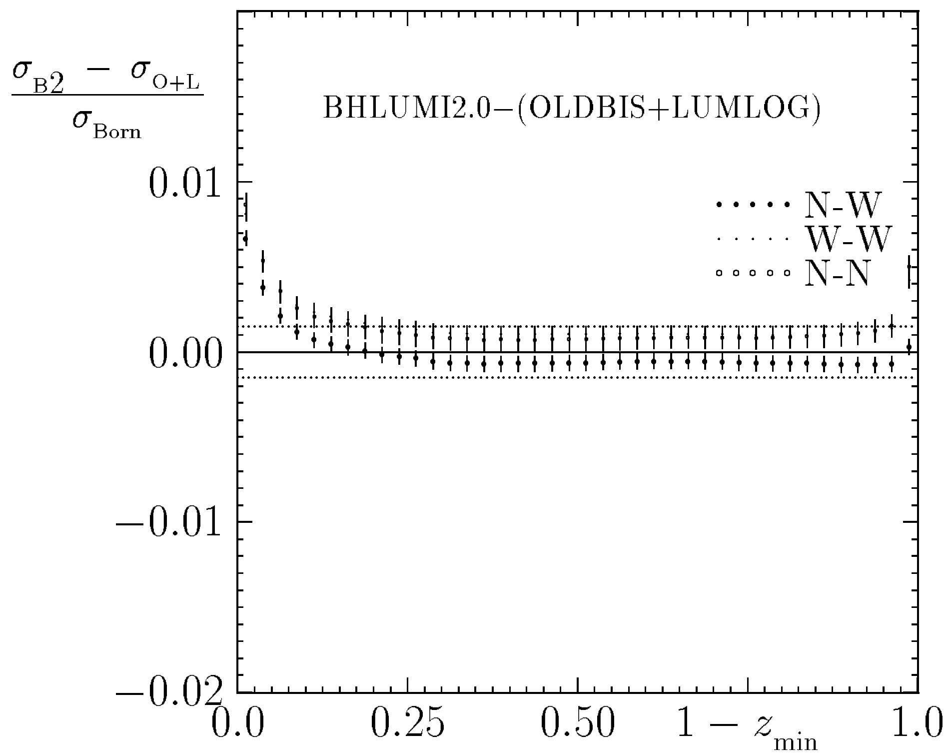


Figure 2: We plot the difference [1] of $\sigma_{\rm B2}$ of BHLUMI 2.0 and $\sigma_{O+L}$ from OLDBIS and LUMLOG. It represents the missing $\mathcal{O}(\alpha^2 L^2)$ bremsstrahlung correction in the BHLUMI version 2.0 event generator [2] together with its technical precision. The difference in the cross-sections (divided by Born) is calculated for the symmetric and asymmetric calorimetric trigger shown in Fig. 3 as a function of the energy cut $z_{\rm min}$. Dotted lines mark the 0.15% limit. Vacuum polarization, $Z$ and $s$ channel $\gamma$ are switched off.

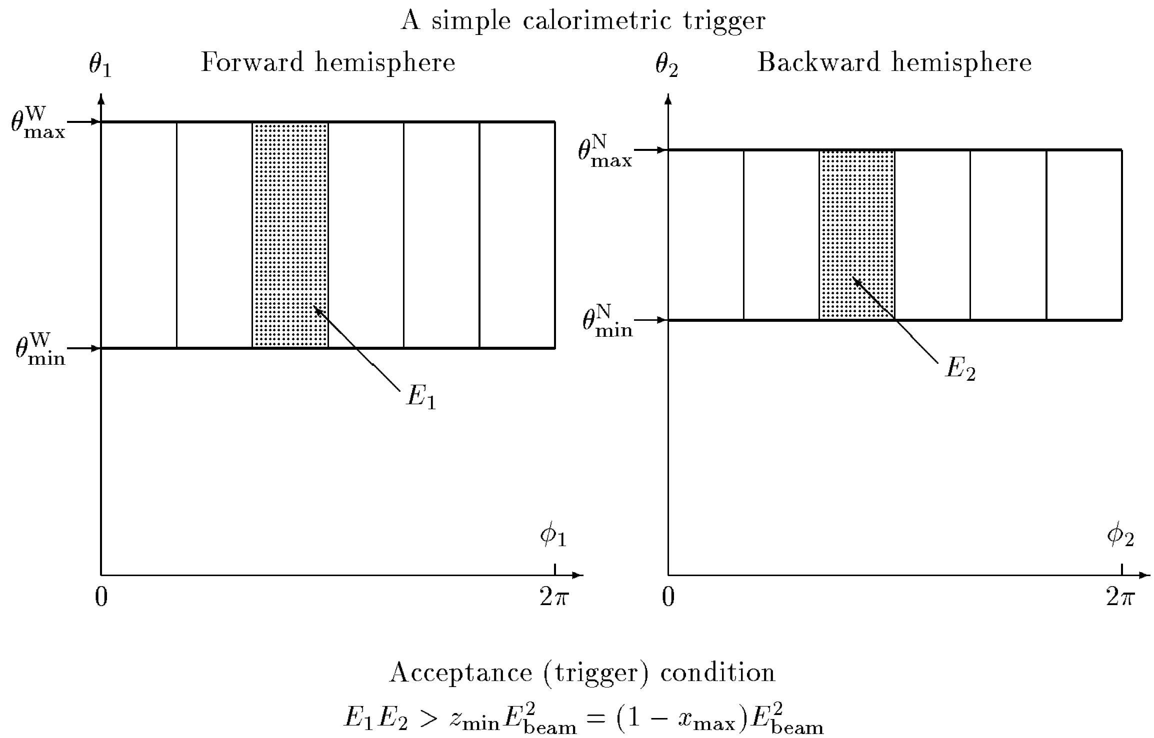


Figure 3: Geometry and acceptance of the simple calorimetric luminosity detector similar to LCAL in the ALEPH experiment. Each side of the detector consists of six calorimetric blocks. A single block measures the total energy of electrons and photons. For accepted events it is required that at least one pair of back-to-back blocks (two shaded blocks in the plot) have enough energy to fulfill the condition $E_1E_2 > z_{\rm min}E^2_{\rm beam} = (1-x_{\rm max})E^2_{\rm beam}$. Shown is the asymmetric type of angular acceptance; thick lines limit wide/narrow $\theta$-range for forward/backward hemispheres.

experimental error. In view of the above, experimentalists have always preferred using the multiphoton Monte Carlo generator BHLUMI 2.0, which includes only that part of $\mathcal{O}(\alpha^2L^2)$ generated by a Yennie–Frautschi–Suura exponentiation (providing excellent realistic differential distributions), and then to employ the OLDBIS plus LUMLOG hybrid solution in order to estimate the missing $\mathcal{O}(\alpha^2L^2)$ correction. For realistic cuts this correction has turned out to be small — typically below 0.2%.

The obvious development path of the above calculation scheme was the following:

a) To implement the $\mathcal{O}(\alpha^2L^2)$ missing part of the matrix element in the multiphoton exponentiated subgenerator of BHLUMI2.0.

b) To provide a new, independent analytical cross-check of the new matrix element.

c) To improve the estimate of the next dominant bremsstrahlung-type corrections — i.e. of $\mathcal{O}(\alpha^2L)$ and $\mathcal{O}(\alpha^3L^3)$ corrections.

d) To again estimate other higher-order corrections like light pairs, vacuum polarization, remnant of $s$-channel $Z$-exchange etc.

In Section 2 we discuss step (a) and in Section 3 step (b). Step (c) is mainly covered in Section 4, but some preliminaries will be given in the preceding section. Point (d), and to some extent also point (c), have already been elaborated [5] in the literature.

Before we get down to more detail, let us explain/define several useful concepts, approximations and terms typical for QED calculations of the LABH cross-section, which will be used or referred to in the course of this contribution.

*Up-down interference:* At angles below 100 mrad all 'photonic' corrections in which the additional photon line connects the upper electron line with the lower positron line — the so-called up-down interference — are strongly suppressed. This phenomenon, which we call 'suppression of the up-down interference', was conjectured and proved numerically, using an $\mathcal{O}(\alpha)$ calculation [3]. In all presented calculations we exploit this approximation.

*The Monte Carlo* technique, used heavily in all presented work, is nothing more or less than the technique of the exact (up to statistical error) integration over the multiparticle phase space. Here we implicitly assume that the differential cross-section is always explicitly written as a product of the matrix element squared, multiplied by the phase–space integration element (this is not true for many QCD calculations).

*The Yennie–Frautschi–Suura (YFS) exponentiation* is a technique of summing exactly all infrared singularities to infinite order that provides us with *exclusive* multiphoton differential distributions. The real hard photons co-exist with the soft photons and there is no need to introduce any additional parameters/cuts to separate them. The multiphoton distributions, with virtual corrections as well, are derived from Feynman diagrams and are calculated/improved order by order. In other words, the $\mathcal{O}(\alpha^n)$ calculation exponentiated in the YFS scheme, when truncated back to $\mathcal{O}(\alpha^n)$, coincides exactly with the ordinary un-exponentiated $\mathcal{O}(\alpha^n)$ calculation. This applies for exclusive multiphoton distributions, and consequently for all inclusive distributions and the total cross-section.

*Trigger* is our short-hand name for the set of kinematical cuts which define accepted events for the LABH total cross-section. Real experimental triggers of LEP/SLC detectors are calorimetric — all photons and electrons satisfying the angular acceptance condition $\theta_{\rm min} < \theta < \theta_{\rm max}$ are registered without any distinction being made among them, and the minimum total energy required is $x = E/E_{\rm beam} > 1 - X_{\rm max}$ in the forward and backward hemisphere simultaneously. In practice, $\theta_{\rm min,max}$ in the forward and backward direction are quite often taken differently (asymmetric trigger). Also, in the real experiment, the association of photons and electrons into a single cluster with energy $E$ and average angle $\theta$ is a little more involved — See Ref. [6] for more details on the various types of experimental triggers. In the calculations we often use as an example the algorithm describing the example of the trigger very close to the new ALEPH SICAL detector. This trigger is defined quite precisely in Fig. 4. It is rather obvious that this kind of trigger cannot be treated analytically and Monte Carlo is in this case the only way to calculate the total cross-section.

## 2 New $\mathcal{O}(\alpha^2)^{\rm exp}_{\rm prag}$ BHLUMI4.0 Monte Carlo

In the following we characterize the new $\mathcal{O}(\alpha^2)_{\rm prag}$ exponentiated matrix element implemented in the new version of Monte Carlo BHLUMI4.0. For reasons of space, we cannot

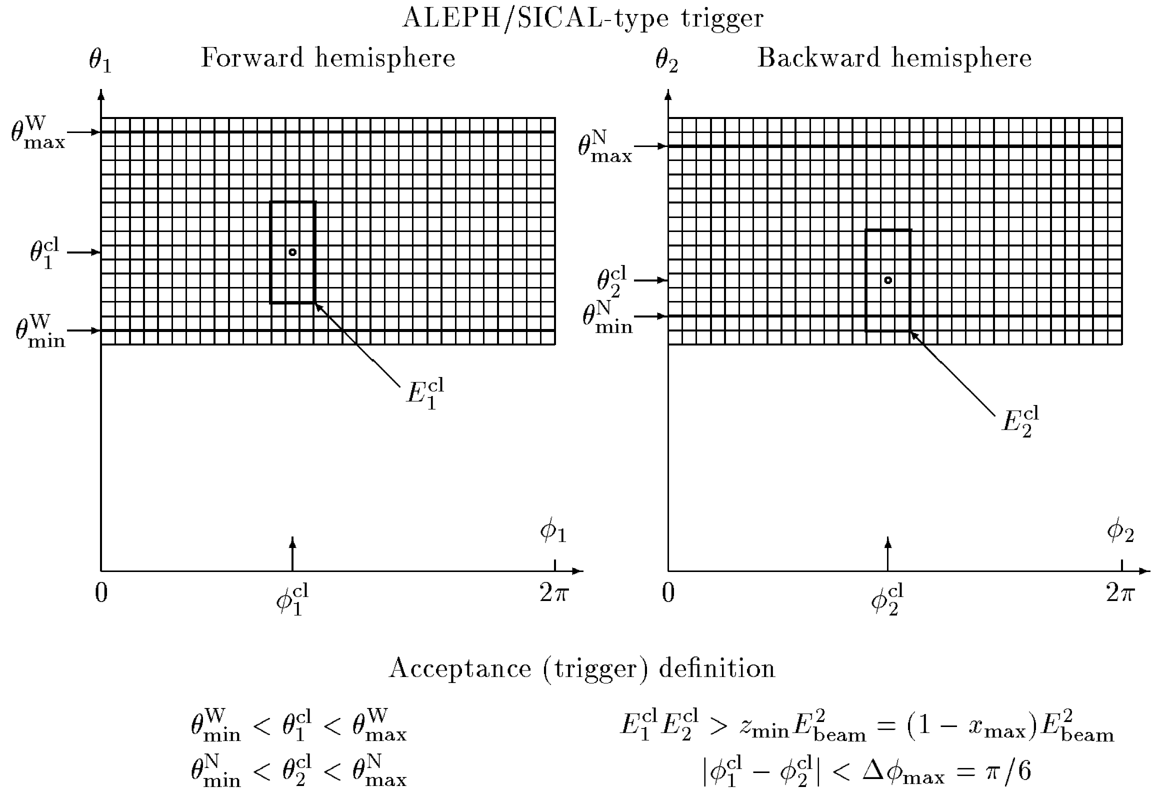


Figure 4: Geometry and acceptance of the SICAL luminosity subdetector in the ALEPH experiment. Each side of the subdetector consists of $16 \times 32$ pads. Single-pad measures the total energy of electrons and photons. A pad of maximum energy and its $3 \times 7$ neighbourhood is called a cluster. Total energy registered in the cluster is denoted as $E_i^{\rm cl}$ and the average position (weighted with energy) is denoted $(\theta_i^{\rm cl}, \phi_i^{\rm cl})$, $i = 1, 2$. By definition $\phi_1 = \phi_2$ for back-to-back particles. We depict asymmetric type of the angular acceptance; thick lines limit wide/narrow $\theta$-range for forward/backward hemispheres. Pads of the cluster which spill over the angular range (thick lines) are used to determine total energy and average position of the cluster (see the example of the backward hemisphere).

include here the full definition of the matrix element used in the program (it would need more than five pages). Nevertheless, we will attempt to characterize all its essential properties.

In $\mathcal{O}(\alpha^2)$, in order to reach the 0.1% physical precision level, it is probably enough to add to the Monte Carlo matrix element, beyond the regular $\mathcal{O}(\alpha)$, the dominant second-order contribution of $\mathcal{O}(\alpha^2 L^2)$ — if possible, with exponentiation. This type of calculation, which we denote as $\mathcal{O}(\alpha^2)_{\rm prag}$, is depicted in Figs. 1a and 5. In fact, in BHLUMI4.0 we include two examples of the $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element, marked A and B, which differ by $\mathcal{O}(\alpha^2 L)$ and $\mathcal{O}(\alpha^2)$ terms. This also is illustrated in Fig. 5. Why are we free to make two choices and why is it profitable? While $\mathcal{O}(\alpha)$ distributions come directly from the Feynman diagrams (no freedom!) the additional $\mathcal{O}(\alpha^2 L^2)$ contributions we derive more simply by twice convoluting the Altarelli–Parisi kernel. The same kind of LL Ansatz was used successfully in the YFS2 and YFS3 Monte Carlo programs [7–9]. For the $\mathcal{O}(\alpha^2 L^2)$

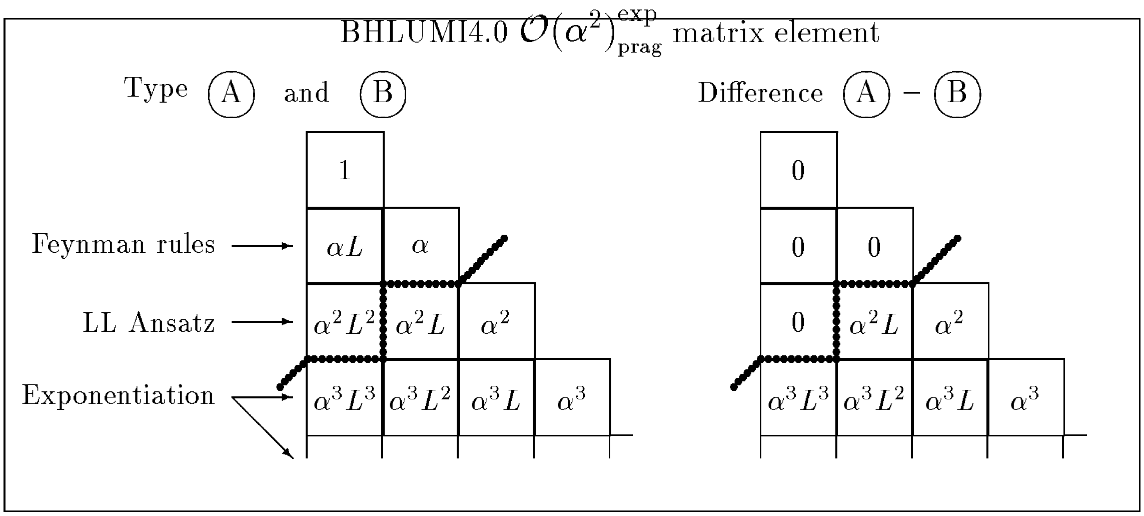


Figure 5: Perturbative content of the matrix element of BHLUMI version 4.0. The correct/complete contributions of $\mathcal{O}(\alpha^2)_{\rm prag}$ are above the dotted line. The diagram right illustrates perturbative content of the difference of the two types A and B of the $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix elements.

contribution, the soft limit is improved by hand[2]. The well-known behaviour and the finite transverse momenta of photons are introduced in the distributions using the soft limit as a model. Obviously, the above procedure has some freedom in the construction of the matrix element, but how much? Let us first note that (even without exponentiation!) the above procedure creates some *non-zero* contributions of $\mathcal{O}(\alpha^2 L^1)$ and $\mathcal{O}(\alpha^2 L^0)$. The two types of $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix elements may have different such contributions. The important advantage of our $\mathcal{O}(\alpha^2 L^2)$ Ansatz is that it is simple, quick in the computer evaluation, and that its LL content — of primary interest — is explicit and therefore very easy to control. As we have already mentioned, the LL Ansatz for the $\mathcal{O}(\alpha^2 L^2)$ comes before the YFS exponentiation. Exponentiation introduces new non-zero contributions of $\mathcal{O}(\alpha^3)$ and higher orders. Among them, the $\mathcal{O}(\alpha^3 L^3)$ contribution will be numerically dominant. Since YFS exponentiation is well founded physically, these higher-order terms substantially improve the perturbative convergence of the calculation — as was proven explicitly for the $\mathcal{O}(\alpha^3 L^3)$ terms [10].

Note that option B for the matrix element, degraded to $\mathcal{O}(\alpha)_{\rm prag}$ exponentiated, is identical to the matrix element in the published BHLUMI2.0 program. BHLUMI4.0 is also backward compatible with BHLUMI2.0 for the OLDBIS and LUMLOG subgenerators, and they are still included. In fact the LUMLOG generator with LL matrix element (exponentiated and unexponentiated) up to the third-order is extended a little, because emission of photons is now included not only in the initial but also in the final state. It is done as previously in the zero transverse-momentum approximation. For completeness we depict the perturbative content of the LUMLOG and OLDBIS event generators in Fig. 6.

[2] In the LL approximation the correct soft limit is not reproduced in the general case.

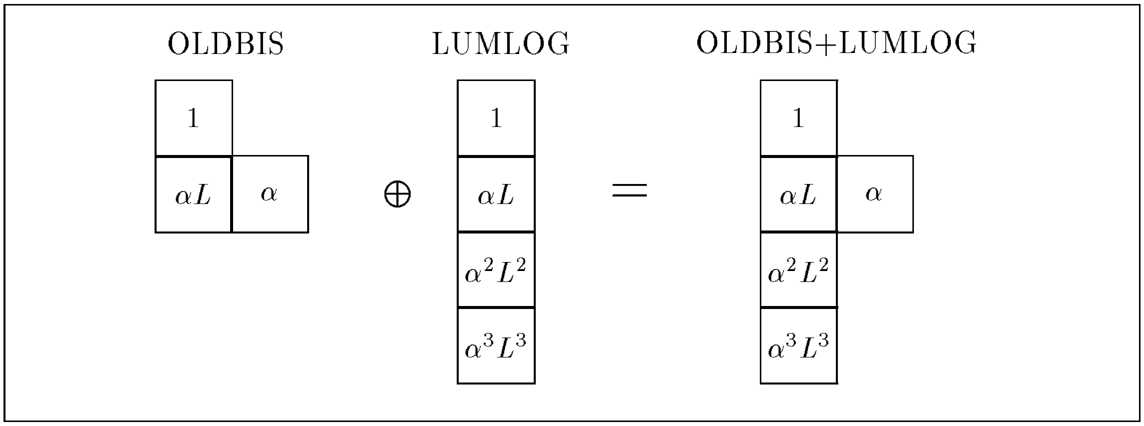


Figure 6: Perturbative content of the matrix element of the hybrid Monte Carlo calculation OLDBIS plus LUMLOG.

# 3 Technical precision

Generally speaking, technical precision is obtained by doing two technically very different calculations and taking the difference. The most powerful method is to take the difference between the Monte Carlo and analytical calculation [4]. The serious disadvantage of this method is that it can be applied only for certain, rather simple kinds of trigger. In the case of BHLUMI4.0 even the existence of one such trigger for which the above method can be applied is of no small importance! The method by which two different Monte Carlo calculations are compared can be applied for a wider family of cuts. Its very serious disadvantage is that if one encounters — as is always the case — an intolerably big difference between the two Monte Carlo results, then it is very difficult, often impossible, to find the source of the difference (debug the corresponding Monte Carlo programs).

We are in the process of determining the technical precision of BHLUMI4.0 using an elaborate multistep method. The steps are:

1. Invent an 'academic trigger' for which *analytical* integration of the BHLUMI4.0 cross-section is feasible down to a precision of $3\times10^{-4}$. This precision level requires a calculation of $\mathcal{O}(\alpha^3)_{\rm prag}$.

2. Perform an analytical calculation and debug the BHLUMI4.0 Monte Carlo program and the corresponding semianalytical calculation/program until $|\sigma_{\rm MC}-\sigma_{\rm analyt}| < 3\times10^{-4}$ is obtained for the 'academic trigger'.

3. Do the same in the (easier) cases of OLDBIS and LUMLOG.

4. Take the difference of BHLUMI4.0 minus (OLDBIS plus LUMLOG) and explore it analytically and numerically in every detail down to a $3\times10^{-4}$ precision.

5. Do an 'adiabatic transition' from the academic trigger to the realistic trigger using a series of intermediate triggers and looking carefully at the evolution of the differ-

ence BHLUMI4.0 minus (OLDBIS plus LUMLOG). It should be understood to a precision comparable to $3\times10^{-4}$ — better than $5\times10^{-4}$, for example.

In the above scenario we take advantage of the extremely important fact that the technical precision for (OLDBIS plus LUMLOG) is practically zero for any kind of trigger!

In the following we will show results from step 1 in the method outlined above: (a) define a set of 'academic' cuts used for the semianalytical integration of the above matrix element over multiphoton phase space, (b) briefly characterize methods used in the analytical integration and the class of corrections kept in the analytical phase space integration (not the same as in the matrix element), and (c) show the numerical agreement of the Monte Carlo (including the new matrix element) with the semianalytical formula down to the 0.03% level (technical precision).

*Ad (a)*: The most important criterion used to define our set of kinematical cuts for semianalytical integration of the $\mathcal{O}(\alpha^2)_{\rm prag}$ new matrix element over the multiphoton phase space (the so-called 'academic trigger') is that this semianalytical integration is at present very feasible. We define the cuts of our 'academic trigger' as: $|t_{\rm min}|<|t|<|t_{\rm max}|$ and $V<V_{\rm max}$, where $t$ is the four-momentum transfer squared and the variable $V$ represents some kind of measure of the total energy carried away by all emitted real photons. We require that $0<V<1$ represent the condition of completeness of the phase space and $V<\epsilon$ the condition that all photons are soft. The $V$-variable we actually define, in terms of the four momenta, as:

$$V = 1-\frac{2(p_1p_2)\;|t|}{[2(p_1p_2)\;+2(p_1K_p)]^2}\;\frac{2(q_1q_2)\;|t|}{[2(q_1q_2)\;+2(q_1K_q)]^2}\;, \tag{1}$$

where $p_i=1,2$ are the four momenta of the incoming and outgoing electron, $q_i=1,2$ are the four momenta of the incoming and outgoing positron and, $K_p$ and $K_q$ are the total of the four momenta of all photons emitted from electron and positron lines.

*Ad (b)*: With the above definition of the phase space window, it is rather straightforward to integrate the $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element, keeping all terms within the $\mathcal{O}(\alpha^2)_{\rm prag}$ approximation. This we found insufficient for the purpose of establishing a technical precision at the 0.03% level because some terms beyond $\mathcal{O}(\alpha^2)_{\rm prag}$ — especially for partially incomplete results — are of that order. We have therefore decided to follow the integration of the $\mathcal{O}(\alpha^3)_{\rm prag}$ approximation; see also Fig. 1b. It means that terms of $\mathcal{O}(\alpha^2L)$, due to our LL Ansatz, and terms of $\mathcal{O}(\alpha^3L^3)$, due to exponentiation, are integrated analytically over the phase space (with the academic trigger) exactly!

The resulting integrated cross section is not very complicated and reads as follows:

$$\begin{aligned}
&\sigma_B^{(2)}(t_{\rm min},t_{\rm max},V_{\rm max}) = \int_{t_{\rm min}}^{t_{\rm max}} dt \int_0^{V_{\rm max}} dV \;\;\rho^{(2)}_{\rm tot}(t,V)\\
&\rho^{(2)}_{\rm tot}(t,V) = b_0\;F(2\gamma)\;e^{2\Delta_{\rm YFS}(\gamma)}\;2\gamma V^{2\gamma-1}\Big\{1+\gamma+\gamma^2/2\Big\}\\
&\quad + b_0\;F(2\gamma)\;e^{2\Delta_{\rm YFS}(\gamma)}\;V^{2\gamma}\Big\{\gamma(-2+V)+\frac{\alpha}{\pi}\ln\,(1-V)(-4+4V-2V^{-1})\\
&\quad + \gamma^2(-2)+\gamma^2\ln\,(1-V)(3-3V/2-2V^{-1})\\
&\quad + \gamma^3(-7V/4)+\gamma^3\ln\,(1-V)[5/4+V/2-2V^{-1}]
\end{aligned}$$

$$
\begin{aligned}
&+ \gamma^3 \ln\,(1-V)^2[-5/8+5V/16+(1/4)V^{-1}] + \gamma^3 \mathrm{Li}_2(V)(2-V)\\
&+ \gamma\frac{\alpha}{\pi}[1/4+11V-(13/4)(2-V)^{-1}+(1/2)(2-V)^{-2}-6(2-V)^{-3}+2(1-V)^{1/2}]\\
&+ \gamma\frac{\alpha}{\pi}\ln\,(1-V)[39/4-19V/4-2V^{-1}\\
&\qquad\qquad -2(2-V)^{-1}+(2-V)^{-2}-(1/2)(2-V)^{-3}-(3/2)(1-V)^{1/2}]\\
&+ \gamma\frac{\alpha}{\pi}\ln\,(1-V/2)[-9/2+3V/4-4(2-V)^{-1}+2(2-V)^{-2}-4(2-V)^{-3}]\\
&+ \gamma\frac{\alpha}{\pi}\ln\,(1-V)^2[19/8-41V/16+V^{-1}] + \gamma\frac{\alpha}{\pi}\ln\,(1-V)\ln(2-V)(-1/2+V/4)\\
&+ \gamma\frac{\alpha}{\pi}\ln\,(1-V)\ln\,(V)(12-10V) + \gamma\frac{\alpha}{\pi}\ln\,(1-V)\ln\,(V/2)(-6+5V)\\
&+ \gamma\frac{\alpha}{\pi}\ln\,(1-V)\ln\,[1-(1-V)^{1/2}](-6+5V)\\
&+ \gamma\frac{\alpha}{\pi}\ln\,(2-V)\ln\,(1-V/2)(3/2-11V/4)\\
&+ \gamma\frac{\alpha}{\pi}\ln\,(1-V/2)^2(3/4-5V/8) + \gamma\frac{\alpha}{\pi}\mathrm{Li}_2(1/2)(-3/2+11V/4)\\
&+ \gamma\frac{\alpha}{\pi}\mathrm{Li}_2[(1-V)/(2-V)](1/2-V/4) + \gamma\frac{\alpha}{\pi}\mathrm{Li}_2[1/(2-V)](1-5V/2)\\
&+ \gamma\frac{\alpha}{\pi}\mathrm{Li}_2\{-V/[2(1-V)]\}(6-5V) + \gamma\frac{\alpha}{\pi}\mathrm{Li}_2[1-(1-V)^{-1/2}](6-5V)\\
&- \xi\gamma\chi(V)/(1-V)\Big\}
\end{aligned}
\tag{2}
$$

where $\gamma = 2(\alpha/\pi)(L-1)$, $b_0 = \chi(\xi)$, $\chi(x) \equiv [1+(1-x)^2]/2$, $\xi = |t|/s$, $F(x) \equiv \exp\,(-Cx)/\Gamma(1+x)$, and $\Delta_{\rm YFS}(\gamma) = \gamma/4 - (\alpha/\pi)(1/2+\pi^2/6)$. Note that the $\mathcal{O}(\alpha^2)_{\rm prag}$ part of the formula is very compact and that its LL content is identical to the non-singlet, second-order, structure function of the photon in the electron[3]. The terms of $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$, which represent most of the formula, finally turn out to be numerically small — in fact below 0.04%.

*Ad (c)*: In Fig. 7a we show a comparison of Eq. (2) with the Monte Carlo BHLUMI4.0. Although the integration over $|t|$ and $V$ is analytically feasible, we do it numerically with the help of the standard Gauss technique: the integrand is a very smooth function, suitable for this method (peaks are removed either by change of variables or subtraction). As we see, the Monte Carlo and semianalytical result differ by up to 0.03%. We conclude that for the above 'academic trigger' we have obtained a 0.03% technical precision — i.e. we have attained the goals of steps 1 and 2.

In the multistep procedure outlined above we have gone through steps 1 to 4 and have the first numerical results for step 5. In step 3 we obtained the simple semianalytical results $\sigma_O^{(1)}$ and $\sigma_L^{(3)}$, which agree with the corresponding Monte Carlo OLDBIS and LUMLOG for 'academic trigger' to better than 0.02%! The next numerical result, shown in Fig. 7b, is relevant to step 4 and demonstrates that the step is completed. As we see, the difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) is under control down to 0.05% because the Monte Carlo and analytical results agree at this precision level. In the next section we further discuss the result of Fig. 7b.

[3]This is due to the fact that the variable $V$ in the LL has a very simple meaning.

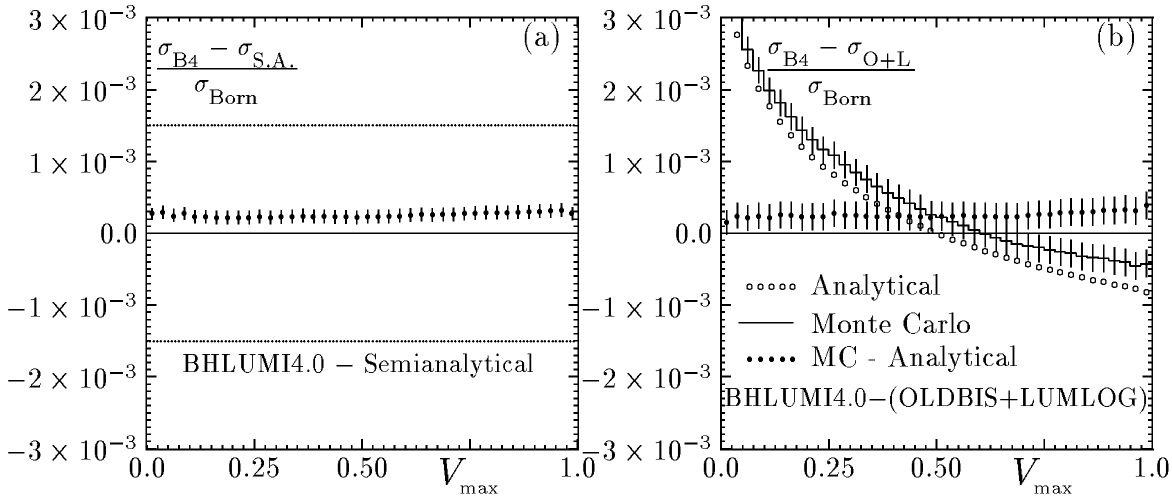


Figure 7: In (a) we show a comparison of BHLUMI4.0 (unpublished) with a semianalytical formula for the 'academic' trigger defined with $|t_{\rm min}| < |t| < |t_{\rm max}|$ and $V < V_{\rm max}$. Dotted lines mark the 0.15% limit. In (b) we demonstrate the same kind of comparison — analytical versus Monte Carlo — for the difference of the cross-section (divided by Born) from three Monte Carlo calculations BHLUMI4.0 minus (OLDBIS plus LUMLOG). The limits $t_{\rm min,\,max}$ correspond to an angular range $26.125 < \theta < 55.875$ mrad.

The crucial question now is whether we can extend this result to triggers other than our 'academic trigger' — in particular whether we can port our result to realistic experimental triggers. This part — step 5 — is still under development and we cannot show all relevant partial results due to space restrictions.

# 4 Physical precision

The precision estimate in our previous work [1] was based to a large extent on the calculation of the difference BHLUMI2.0 minus (OLDBIS plus LUMLOG), which for BHLUMI2.0, represented the missing $\mathcal{O}(\alpha^2 L^2)$ plus technical precision. With the new matrix element in BHLUMI4.0, which includes the complete $\mathcal{O}(\alpha^2 L^2)$ contribution, we look immediately into the same three-generator difference. The result of such a comparison, with the scale on the vertical axis inflated by a factor of almost ten, is presented in Fig. 8. It is done for the same trigger type and angular range as in the earlier publication, calorimeter of the ELCAL/ALEPH type [1], and also for the new ALEPH/SICAL-type detector at a lower angular range.

At first sight, the new result in Fig. 8 looks completely compatible with the earlier published result of Fig. 2, with the difference again well within 0.15%. Of course, the interpretation of this difference is not the same now as previously. The multiphoton Monte Carlo BHLUMI4.0 includes $\mathcal{O}(\alpha^2 L^2)$ corrections, hence the plotted difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) is potentially dominated by the $\mathcal{O}(\alpha^2 L)$, $\mathcal{O}(\alpha^3 L^3)$ and technical precision. (The estimate of the technical precision from Fig. 7 does not apply

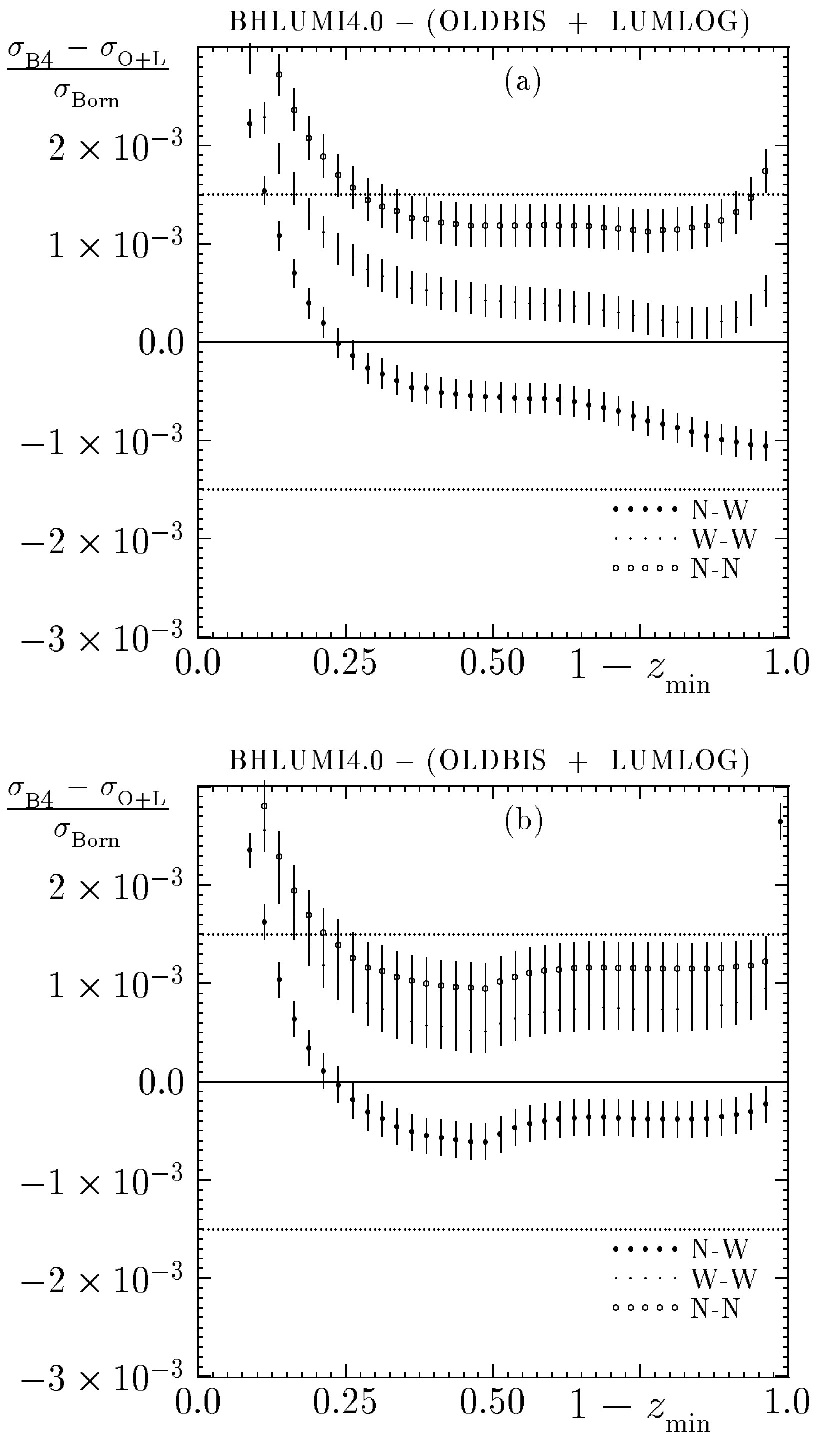


Figure 8: We plot the difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) (a) for the ALEPH/ELCAL type calorimeter/trigger (angular range 60–120 mrad) as previously defined [1] and (b) for ALEPH/SICAL detector (25–60 mrad). Dotted lines mark the same 0.15% limit as in Fig. 2.

here automatically, due to the different type of trigger.) The $\mathcal{O}(\alpha^2 L^2)$ is absent from the new results of Fig. 8. These new results are very encouraging but should be treated as *preliminary* — they will soon be subjected to a new round of tests.

In the following we present additional numerical results that will allow us to better understand the meaning of the results in Fig. 8. The immediate question to answer is: How big was the missing $\mathcal{O}(\alpha^2 L^2)$ correction in the published version of BHLUMI2.0? For the answer, we need only look at the difference BHLUMI4.0 minus BHLUMI2.0. Since the published BHLUMI2.0 includes the matrix element of type A, we examine the difference $\mathcal{O}(\alpha^2)^{\rm exp}_{\rm prag,A} - \mathcal{O}(\alpha^1)^{\rm exp}_{\rm prag,A}$. It is plotted in Fig. 9a for the SICAL detector of ALEPH. We see that the result is small in comparison with the typical values/estimates 0.25%–0.5% quoted in previous papers [4, 1] for the $\mathcal{O}(\alpha^2 L^2)$ corrections. We simply conclude that the choice of the matrix element in BHLUMI2.0 was very lucky! To see it more clearly, we show a similar quantity for the matrix element of type B in Fig. 9b. Here, the situation looks more 'normal'. The missing $\mathcal{O}(\alpha^2 L^2)$ contribution for the experimentally relevant $X_{\max} \simeq 0.5$ is $-0.25\%$. Of course, the next logical question is: Do the two $\mathcal{O}(\alpha^2)_{\rm prag}$ results of type A and B agree? Yes, they do. And, as we see in Fig. 9c, the corresponding difference is tiny indeed — just $\simeq 0.01\%$!

All the above cross-checks, together with the results from the previous section, give us a strong hint that the new $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element in BHLUMI4.0 is correctly implemented. Nevertheless, since the problem of high technical precision for a realistic trigger is still not solved, we say that the difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) in the plots of Fig. 8 represent the missing $\mathcal{O}(\alpha^3 L^3)$, $\mathcal{O}(\alpha^2 L)$ and the technical precision. In fact the $\mathcal{O}(\alpha^3 L^3)$ can be practically eliminated from this list. Using the semianalytical formula (2) we can calculate *exactly* the missing $\mathcal{O}(\alpha^3 L^3)$. The $\mathcal{O}(\alpha^3 L^3)$ missing terms are known [10] and can be easily included in Eq. (2)! The effect of such a modification on the cross-section is shown in Fig. 9c. It is negligible: below 0.02%. This result is not that surprising, because it was shown [10] that the YFS exponentiation sums up the LL part of higher orders very efficiently. (We suspect that it may not hold for the $\mathcal{O}(\alpha^2)$ calculation without YFS exponentiation.) Note that the above result extends for any calorimetric experimental trigger because it is of a pure LL character. We conclude, therefore, that Fig. 8 contains only the missing $\mathcal{O}(\alpha^2 L)$ and technical precision.

In view of the above discussion, can we already improve on the total precision of the luminosity cross-section? First of all, we previously estimated [1] that the $\mathcal{O}(\alpha^2 L)$ contribution is generically of an order of 0.1%. In fact the value 0.09% was used and summed up with the 0.15% estimate of the missing $\mathcal{O}(\alpha^2 L^2)$ from Fig. 2 to obtain a total bremsstrahlung error of 0.24%. Since we now know that the missing $\mathcal{O}(\alpha^2 L^2)$ was absent in the (old) Fig. 2 by chance and in the (new) Fig. 8 by construction, these figures show us $\mathcal{O}(\alpha^2 L)$ and technical precision. The 0.15% spread of the curves in Fig. 8 is very compatible with the generic estimate of 0.1%. But since the generic estimate and the spread of the curves now represent the same thing, in order to avoid double counting, instead of their sum we should take the maximum of the two. This gives us roughly a 0.15% estimate of the *total bremsstrahlung uncertainty* in BHLUMI4.0 — i.e. the corresponding physical and technical precision. This is a net improvement over the published 0.24%. In order to improve on the above we need to have a better, separate, estimate of the technical precision for the *experimental* trigger below 0.05% and a better estimate (by direct calculation) of the $\mathcal{O}(\alpha^2 L)$ missing contribution. Of course, inclusion of the error from the vacuum polarization, light fermion pairs, etc., will increase the error, but not by too much. This more detailed analysis will be presented elsewhere.

The central question of the physical precision of the BHLUMI4.0 problem is now obviously the following: How big is the missing second-order subleading $\mathcal{O}(\alpha^2 L)$ correction? It would be best to calculate this object *directly* and there are attempts in this direction by the authors and others [11]. For the moment let us gather all *indirect* information on this subject. On the one hand, the earlier [1] estimate of 0.1% is in good agreement with the 0.15% variation in Fig. 8. On the other hand, there is some indication that the actual value may be smaller. This comes mainly from the analytical inspection of the difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) for the academic trigger — see also Fig. 7b — and confirmed to some extent by Fig. 9c for the real experimental trigger.

**Table 1**

The canonical coefficients indicating the generic magnitude of various leading and subleading contributions up to third order. The big log $L$ is calculated for $\theta = 25$ mrad.

| Canonical coefficients | | | | |
|---|---|---|---|---|
| | non-calorimetric case | | calorimetric case | |
| $\mathcal{O}(\alpha L)$ | $\frac{\alpha}{\pi}4L$ | $140\times10^{-3}$ | $\xrightarrow{1/2}$ | $70\times10^{-3}$ |
| $\mathcal{O}(\alpha)$ | $2\frac{1}{2}\,\frac{\alpha}{\pi}$ | $2.5\times10^{-3}$ | $\xrightarrow{1}$ | $2.5\times10^{-3}$ |
| $\mathcal{O}(\alpha^2 L^2)$ | $\frac{1}{2}\left(\frac{\alpha}{\pi}4L\right)^2$ | $10\times10^{-3}$ | $\xrightarrow{1/4}$ | $2.5\times10^{-3}$ |
| $\mathcal{O}(\alpha^2 L)$ | $\frac{\alpha}{\pi}\left(\frac{\alpha}{\pi}4L\right)$ | $0.35\times10^{-3}$ | $\xrightarrow{1/2}$ | $0.2\times10^{-3}$ |
| $\mathcal{O}(\alpha^3 L^3)$ | $\frac{1}{3!}\left(\frac{\alpha}{\pi}4L\right)^3$ | $0.45\times10^{-3}$ | $\xrightarrow{1/8}$ | $0.05\times10^{-3}$ |

The closer look into $\mathcal{O}(\alpha^2 L)$ terms in the analytical formula for BHLUMI4.0 minus (OLDBIS plus LUMLOG) reveals that almost all such terms are numerically unimportant because the value of the coefficient $(\alpha/\pi)^2 4L = 0.035\%$ is small. All terms which do matter numerically have special reasons for this. They are bigger because they include enhancement factors which can be understood and traced back to certain peculiarities of the calculation. In the case of Fig. 7b all these enhancement factors reflect either a lack of exponentiation in OLDBIS or a zero-transverse-momentum approximation in LUMLOG. (They are typically logarithms of cut-off parameters and $\zeta_2 = \pi^2/6$ in the virtual corrections.) The fact that the result shown in Fig. 7b is bigger than the canonical 0.035% reflects these artefacts in the OLDBIS plus LUMLOG and not the overlooking of such contributions in BHLUMI4.0. The important lesson for future evaluation of $\mathcal{O}(\alpha^2 L)$ contributions is that any such contribution above 0.035% has to be checked and explained

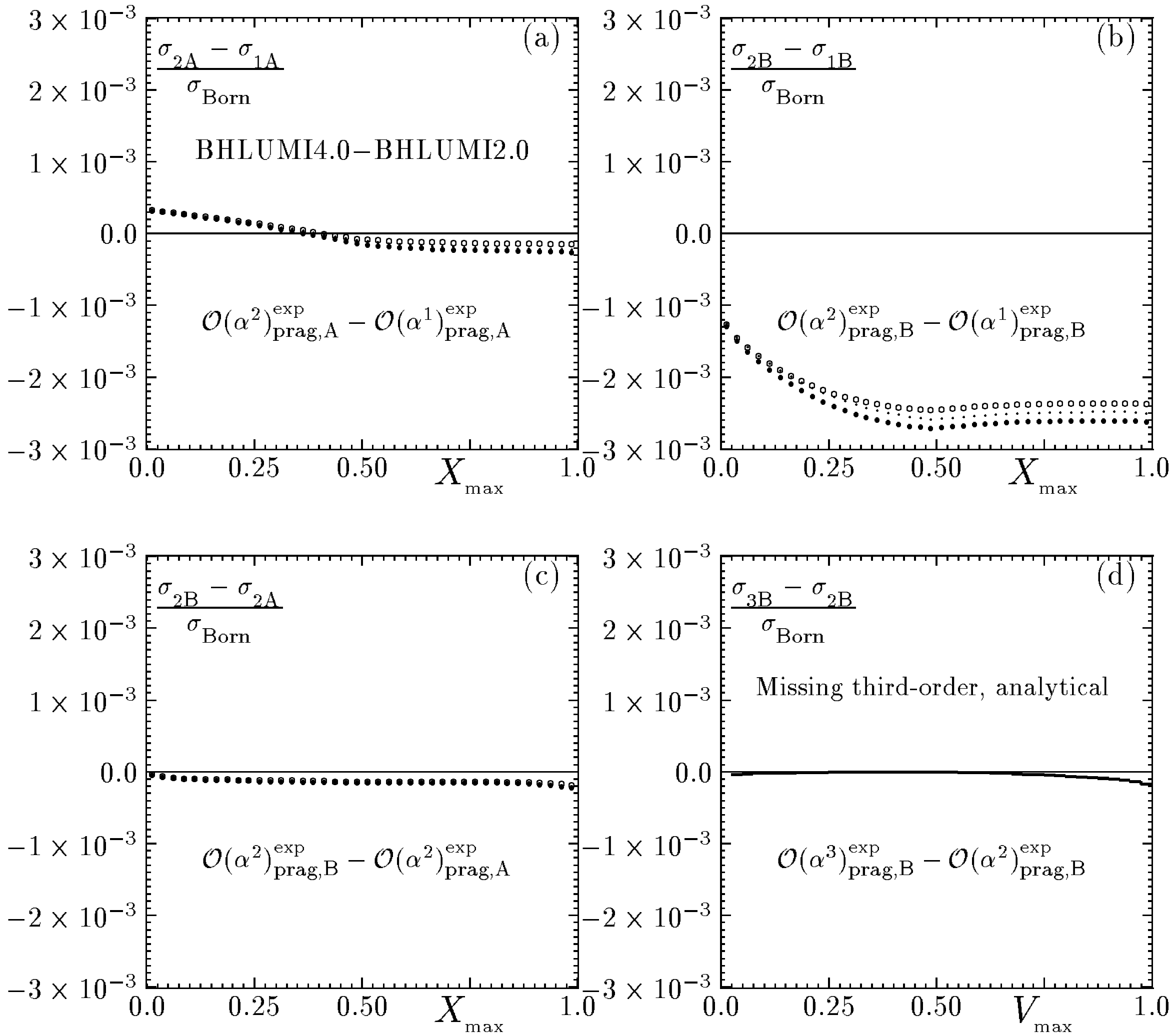


Figure 9: Plots (a)–(c) show Monte Carlo results for the SICAL/ALEPH detector and (d) shows analytical result for the 'academic' trigger. All cross-sections are divided by the Born value and plotted as a function of the energy cut $X_{\max}$ or $V_{\max}$. In (a) we demonstrate the 'missing second-order' of the already-published BHLUMI2.0 as the difference between it and the new version, BHLUMI4.0. In (b) we plot the same quantity for matrix element type B. Plot (c) demonstrates the difference in results between A and B type $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix elements. In (c) we show for the 'academic' trigger how the cross-section would change if the matrix element B were upgraded to $\mathcal{O}(\alpha^3)_{\rm prag}$.

separately because it needs a special reason — the enhancement factor — to exist! We have done the above exercise a little more systematically and, looking into coefficients in the Eq. (2), we have obtained all typical 'canonical' coefficients for various leading and subleading contributions up to the third-order, these being included in Table 4. Of course, the coefficients are smaller for calorimetric detection of the final-state electrons and we tried also to estimate this effect. Note that the canonical coefficients in Table 4 agree very well with: for $\mathcal{O}(\alpha^2 L^2)$, Fig. 9b; for $\mathcal{O}(\alpha^2 L)$, Fig. 9c; and for $\mathcal{O}(\alpha^3 L^3)$, Fig. 9d.

# 5 Summary and outlook

The status of our work on the QED corrections for the luminosity cross section can be summarized as follows:

- The $\mathcal{O}(\alpha^2)_{\rm prag}$ exponentiated matrix element in the new version of BHLUMI4.0 is implemented and we have a lot of high-precision (technical precision: $3 \times 10^{-4}$) evidence that it was done correctly.
- A technical precision as high as $3 \times 10^{-4}$ has been established for BHLUMI4.0 for the special 'academic' type of trigger (cut-offs).
- The OLDBIS/LUMLOG tandem is used to port the above high technical precision to realistic examples of triggers under development.
- There are indications that the main $\mathcal{O}(\alpha^2 L)$ contribution to physical precision is below 0.1% — even 0.03% possibly — but we have to stick to a conservative estimate of 0.1%, which provides us with the new BHLUMI4.0 total bremsstrahlung uncertainty of 0.15% for the generic experimental trigger. This is a considerable improvement over the previously published value of 0.24%.

In our future work we plan to extend the technical precision $3 \times 10^{-4}$ to truly realistic experimental triggers, implementing the $\mathcal{O}(\alpha^2 L)$ part of the matrix element or providing a more solid numerical evaluation/estimate of its magnitude. Realization of the above will definitely allow total precision to be brought below 0.1%, which will match the best experimental errors.

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